An algorithm for the MMP in dim 3
(arXiv: 2603.13703)

§ Intro

MMP (char. 0, dim 3, $\mathbb{Q}$-factor. & log term. sings)
Given a proj. var X, keep transforming it until we get nef K_X or a Mori fiber space by doing:

1) Contract a K-negative extremal ray;
2) If we get a flipping contr, we take the flip.

$\rightsquigarrow$ Output: minimal model (nef K_X)
or
Mori fiber space

Q. How to compute/perform each step algorithmically?

Main Thm $\exists$ algorithm that performs the MMP in dim 3 over $\overline{\mathbb{Q}}$.

Rem
- Elements of $\overline{\mathbb{Q}}$ are represented by finite data. Arithmetic operations are computable.
  Similarly for polynomials / $\overline{\mathbb{Q}}$.
- Inputs and outputs of the algorithm are given by finitely many polynomials / $\overline{\mathbb{Q}}$.

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§ Data types

$k = \overline{\mathbb{Q}}$

Def ① monograded variety
:= proj var written as $\mathrm{Proj}\ \dfrac{k[x_0, \dots, x_n]}{(f_1, \dots, f_e)}$ ($\mathbb{N}$-graded)
w/. $\deg(x_i) = c_i \in \mathbb{Z}_{>0}$
f_j : (weighted) homogeneous

part of data

② bigraded variety

:= proj var written as $\mathrm{Proj}\ \dfrac{k[y_0,\cdots,y_n,x_0,\cdots,x_m]}{\underbrace{(h_1,\cdots,h_r)}_{\mathbb{N}^2\text{-graded}}}$

w/. $\deg(y_j) = (d_j, 0)$, $d_j \in \mathbb{Z}_{>0}$

$\deg(x_i) = (0, c_i)$, $c_i \in \mathbb{Z}_{>0}$

h_j : bihomog. polynomials.

mono. var $\subset \mathbb{P}(c_0,\cdots,c_n) = \mathbb{P}(\underline{c})$

bi. var $\subset \mathbb{P}(\underline{d}) \times \mathbb{P}(\underline{c})$

__Def__ ① $Y \subset \mathbb{P}(\underline{d})$, $X \subset \mathbb{P}(\underline{c})$ monograded varieties

A graph mor. $f : Y \to X$ is a mor. given by

the graph $\Gamma_f \subset Y \times X \subset \mathbb{P}(\underline{d}) \times \mathbb{P}(\underline{c})$ as a bigraded var.

② Y, X : monograded

$W \subset Y \times X$: bigraded

$p_X|_W : W \to X$ ← B2M projection

(used for · Stein factor.
- flip)

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§ Algorithm

MMP : based on the analysis of the Mori cone, which lives in a vec. sp of dim $\rho(X)$ (Picard number)

__Difficulty__ No known algo. for computing the Picard number.

(Poonen-Testa-van Luijk : algorithm under the Tate conj.)

Algorithm

Input: a monograded variety X of dim 3, over $\overline{\mathbb{Q}}$, w. only normal $\mathbb{Q}$-fact. & log term. sings

Output: an MMP sequence starting from X s.t. varieties & morphisms in the sequence are given as monograded vars and graph morphisms, respectively.

Procedure (Outline):

① If K_X is nef $\Rightarrow$ STOP

② Compute the Stein fac. of $\phi_{|E|}$ for every base-pt-free divisor E, until we find the contr. mor of some K-negative extremal ray.

(Note: $\overline{\mathbb{Q}}$ countable.)

③ If it is
- Mori fib sp $\Rightarrow$ STOP
- div. contr $\Rightarrow$ Go back to ①
- flipping contr $\Rightarrow$ Compute the flip & go back to ①.

Rem: a large room for improvement in the efficiency.

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§ More details on each step.

Nefness of K_X In general, "not nef" is semidecidable.

From the abundance conj (valid in dim 3)

K_X nef $\Longleftrightarrow$ K_X semiample

semiample is semidecidable.

So, for K_X in dim 3, both "nef" & "not nef" are semidecidable. $\Rightarrow$ "K_X is nef" is decidable.

Contraction $f: Y \to X$ mor of normal proj vars w.
$f_* \mathcal{O}_Y = \mathcal{O}_X$, Y: $\mathbb{Q}$-fact & log term.

Then, f is the contr. of some K-negative extremal ray
iff. ① $R^i f_* \mathcal{O}_Y = 0$ $(i > 0)$
② $b_2(Y) - b_2(X) = 1$ $\quad (b_2(X) = \dim H_2(X, \mathbb{C}))$
③ For some (every) curve C contracted by f, $K_Y \cdot C < 0$.

can be checked algorithmically

Flip $f: Y \to X$ flipping contr.

The flip of f = $\mathrm{Proj}_X \bigoplus_{n \geq 0} \mathcal{O}_X(n \cdot m K_X)$ $\quad (^\forall m > 0)$

If m is sufficiently factorial,

$$\mathcal{S}^m := \bigoplus_{n \geq 0} \mathcal{O}_X(m K_X)^{\otimes n} \twoheadrightarrow \bigoplus_{n \geq 0} \mathcal{O}_X(nm K_X)$$

$$\mathrm{Proj}_X(\quad " \quad) \hookleftarrow \underbrace{\mathrm{Proj}(\quad " \quad)}_{\text{flip}}$$

The main component of $\mathrm{Proj}_X \mathcal{S}^m$ is realized as
the bigraded var $\mathrm{Proj}\ \underbrace{\underbrace{R}_{\text{homog. coor. ring of } X}[I^{(m)}t]}_{\text{(non-symbolic) } K_X + \text{ Cartier. Rees alg.}}$ $\quad$ sym. power of I

Compute $Z^m = \mathrm{Proj}\ R[I^{(m)}t]$ for $m = 1!, 2!, 3!, \ldots$

If Z^m is normal, & if $Z^m \to X$ has excep. locus of codim ≥ 2,
then $Z^m \to X$ is the flip of f.

§ Stein factorization

Key

Prop [Y] $X = \mathbb{P}\mathrm{roj}\, R$ bigraded

M, N : bigraded f.g. R-modules

$\mathcal{M}, \mathcal{N}$: ass. $\mathcal{O}_X$-modules

If $\mathbb{Z}^2 \ni \mathbb{r} \geq$ (pair of explicit numbers),

$$\bigoplus_{\mathbb{v} \in \mathbb{N}^2} \mathrm{Hom}_X(\mathcal{M}, \mathcal{N}(\mathbb{v})) = \mathrm{Hom}_R(M_{\geq \mathbb{r}}, N)_{\geq (0,0)}$$

(bigraded version of [G. Smith '00])

$f : Y = \mathrm{Proj}\, B \longrightarrow X = \mathrm{Proj}\, A$

$\Gamma_f = \mathbb{P}\mathrm{roj}\, R \subseteq \mathbb{P}\mathrm{roj}\, B \otimes A$.

$Y \to Z \to X$ the Stein fac.

$$\Rightarrow Z = \mathrm{Proj}\, \underbrace{\mathrm{Hom}_R(R_{\geq \mathbb{r}}, R)_{0, \geq 0}}_{\text{monograded ring}}.$$

$\rightsquigarrow Z \to X$